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CH.EN HUGENS
né en 1629. mort en 1695.

THE MATHEMATICS TEACHER

Volume XXVI Number 1

Edited by William David Reeve

The Function Concept in Algebra

By W. R. Longley
Yale University

I. The Function Concept

No general movement in education has introduced details which were new to all teachers. When the name of Montessori was in all of the newspapers as the prophet of new and better things in elementary education I recall a letter from a somewhat indignant teacher in a small school in Mississippi asserting that she had always used the identical methods that were making the Italian teacher famous. I have no doubt that many secondary school teachers made use of the function concept years before it was brought to general attention in the report of the National Committee on the Reorganization of Mathematics in Secondary Education. The material in this report was widely circulated, was discussed in meetings all over the country, and quite generally approved before publication. Yet it has failed to produce the maximum beneficial effect. Too many teachers who really approve of the function concept in theory find difficulty in the practical classroom application. Only a few months ago an experienced teacher in one of the large New England high schools referred to the topic as "collegiate." She could perfectly well omit it from the text which she was using, but she wanted it cut out of the text so that she would not have to omit it.

The National Committee and the College Board. The work of the National Committee accomplished important results but it suffered from two handicaps. It lacked authority to enforce any of its ideas and its existence was temporary. Fortunately, to a certain extent at least, these two defects were remedied by the College Entrance Examination Board. There are some undesirable effects of the influence of the Board examinations on secondary school teaching which no one deplores more than I do. But I believe that the beneficial outweigh the deleterious effects and an example is the result of the present definitions of requirements in mathematics. These definitions were almost exactly a restatement of the results reached by the National Committee. The effect was that some schools were forced to follow more modern ideas in their teaching of mathematics. I feel sure that some of our best known schools would now be teaching algebra exactly as it was taught twenty years ago if they had not been compelled to change by the College Board requirements.

Some features of the progress represented in the report of the National Committee are readily enforceable by examinations. An obvious example is the introduction of a certain amount of trigonometry into elementary algebra. Other features are not so easily forced upon an unwilling teacher. While we believe that the function concept enriches the teaching throughout the whole secondary school course, it does not lend itself naturally to examination purposes. Something can and has been done in the examination papers to promote the use of the idea but the maximum benefit to the pupil can not be forced in this way. The College Board defines minimum requirements and we are now interested in the instruction which gives to each individual pupil the maximum benefit.

The function concept of the National Committee. For some reason, which I completely fail to understand, the word function in its technical mathematical sense brings to certain people a vision of insurmountable difficulties. This was apparent when the National Committee report was written. The report says:

> Since this recommendation is peculiarly open to misunderstanding on the part of teachers, it seems desirable to devote a separate chapter to a rather detailed discussion of what the recommendation means and implies. It will be seen in what follows that there is no disposition to advocate the teaching of any sort of function *theory*. A prime danger of misconception that should be removed at the very outset is that teachers may think it is the notation and the definitions of such a theory that are to be taught. Nothing could be further from the

intention of the committee. Indeed, it seems entirely safe to say that the word "function" had best not be used at all in the early courses.

What is desired is that the idea of relationship or dependence between variable quantities be imparted to the pupil by the examination of numerous concrete instances of such relationship. He must be shown the workings of relationships in a large number of cases before the abstract idea of relationship will have any meaning for him. Furthermore, the pupil should be led to form the habit of thinking about the connections that exist between related quantities not merely because such a habit forms the best foundation for a real appreciation of the theory that may follow later, but chiefly because this habit will enable him to think more clearly about the quantities with which he will have to deal in real life, whether or not he takes any further work in mathematics.

Whereas there can be little doubt about the small value to the student who does not go on to higher studies of some of the manipulative processes criticized by the National Committee, there can be no doubt at all of the value to all persons of any increase in their ability to see and to foresee the manner in which related quantities affect each other.

To attain what has been suggested, the teacher should have in mind constantly not any definition to be recited by the pupil, not any automatic response to a given cue, not any memory exercise at all, but rather a determination not to pass any instance in which one quantity is related to another, or in which one quantity is determined by one or more others, without calling attention to the fact, and trying to have the student "see how it works."

These quotations show clearly what the National Committee had in mind, and what most of us now understand by the function concept in secondary teaching. They show equally well that the concept does not readily furnish material for examination purposes. Its use rather is to "open the door to mathematics and give the student that appreciation of the meaning of mathematics which is an essential today for every educated man or woman."

II. When Should the Function Concept Be Stressed?

The report of the National Committee remarks that "instances of the function concept occur in literally thousands of cases in both algebra and geometry." While this might give the impression that the function concept should always be in the foreground, it is equally true that there are thousands of times when the function concept is quite foreign to the topic under discussion. In other situations emphasis on the concept may or may not be advisable. This leads to the question as to when the concept is most important.

Classes of Pupils. To answer this question we must give a consideration to the pupils and the modern tendency to secure a greater de-

gree of adjustment to individual needs. This is one of the penalties of progress. Years ago it would not have confronted us. The time was when the college curriculum consisted of Latin, Greek, and mathematics and everybody took the same course, with the result that many questions were simplified. Prizes and scholarships were awarded on the basis of proficiency in these three subjects. Such prizes, never thought of as shortsighted when they were established, still exist at Yale. At one time every student in the college was eligible. Now there is seldom one Freshman who has taken all three subjects and it has been necessary to redistribute the prizes as much as possible under the terms of the gifts in order to make any awards.

At the present time the question as to who should study mathematics and how much each one should study has to be answered in every stage from elementary school to college and university. We must now consider various classes of students which, for our present purpose, may be limited to the following four.

1. The pure mathematician, who is essentially a logician and philosopher, dealing with ideas and abstractions of his own design.

2. The applied mathematician, who deals with facts as they are and with problems which arise in the physical world. This class includes physicists, chemists, biologists, doctors of medicine, engineers, architects, business executives, and artizans.

3. The educated layman, who is interested in modern thought along many lines in which he takes no active creative part.

4. The non-mathematical specialist, who has intellectual interests confined to fields where mathematics is not essential.

The problem of the education of the pure mathematician was solved many years ago. The solution consisted of a fairly standardized sequence of courses beginning with four years in the secondary school, continuing through four years in college, and ending with three or more years in the graduate school. The sequence was as follows: elementary algebra, plane geometry, solid geometry, trigonometry, college algebra, analytic geometry, calculus. These courses carried the student through junior year in college, after which there was some variation. Further work usually included the theory of equations, solid analytic geometry, advanced calculus, theoretical mechanics, and certain broad, fundamental graduate courses followed by intensive study and research in some special field.

From the beginning of high school to senior year in college there was almost no choice for the student and little variation in different

schools and colleges. This program was laid out by mathematicians for the education of mathematicians and for that purpose has never been severely criticized. Criticism did arise because the scheme was being used for other purposes than the one to which it was especially fitted. Overlooking any defects in the secondary school work, it became evident about twenty years ago that the college offering was not altogether satisfactory. Specific criticism was made by a committee representing various engineering societies and appointed to study engineering education. Its report was published in 1918 by the Carnegie Foundation for the Advancement of Teaching. The substance of the criticism may be stated as follows. The mathematicians have laid out a sequence of courses beginning with the freshman year in college and leading after seven or eight years to the degree of doctor of philosophy. Any student who wishes to take two years of mathematics in college has no choice but to take the first two years of this seven year course. While this may be the very best use of his time for the student who intends to continue the study of mathematics, there is a grave question as to the value for the student, particularly the engineering student, who must limit the time devoted to mathematics.

This criticism is more or less typical of the objections which have brought about changes in both college and secondary school courses during the last twenty years. It is conceivable that if the pure mathematician type of students had existed in sufficient numbers and that if it were possible to segregate them for instruction by themselves, we should still desire to give the training exactly as outlined above. But the predominance of other types of pupils has made it necessary for the early training of the mathematicians to fit in with that of larger groups.

The second class into which we have divided the student body comprises all of those who expect to use mathematics as a tool. Within this group are subclasses such as (1) the artizan who does not go to college, (2) the engineer who attends a technical school for four years, (3) the business man who attends a college of arts for four years, and (4) the scientist and professional man who follows his college course by study in a graduate school. These subclasses present problems differing in detail but having two elements in common. In the first place each has a definite limited amount of time to give to the study of mathematics and the problem is to provide the maximum training for each individual within the time available. In the second place mathematical ideas must be supplemented by a certain com-

mand of technique, without which the mathematical tools can not be used effectively.

The third class, consisting of educated laymen, differs from the class of applied mathematicians chiefly because the mathematical ideas are of prime importance and technique is a secondary consideration. In general the third class gives about half as much time to mathematical courses as the second class.

With the fourth class we are not now concerned. These students specialize in literature, languages, history, music, and art. They wish a minimum mathematical training and I am not prepared to say what that minimum should be.

As the whole body of pupils progresses through school and college there is a natural tendency for the classes to become separated and in this way to simplify the problems of subject matter and methods of teaching. We are making some progress toward assisting this natural tendency to operate earlier by the use of prognostic tests. But until these tests are more generally relied upon, we are still confronted with the task of teaching the same course in the same classroom to groups with different characteristics. The necessity of doing this at Yale has led to certain principles and methods of which I can speak from personal experience.

Experience at Yale. Prior to 1920, Yale College and the Sheffield Scientific School were distinct institutions. The undergraduate courses were given by separate corps of teachers and the problems of one school had no effect upon those of the other. There had been a common tendency in the early courses to cut material from analytic geometry and to begin the calculus earlier but this was not due to concerted action. Neither group of instructors paid any attention to what the other group was doing.

In 1920 there was a reorganization which involved the establishment of a Common Freshman Year through which students of both Yale College and the Sheffield Scientific School should pass. At that time it seemed of prime importance to make no distinction between the members of the freshman class on the basis of their future choice of school. In particular this meant that instruction in mathematics, or any other subject, should not be made to depend on the future career of the student. There was, and still is, some difference in the course offered to students who entered with the minimum preparation of elementary algebra and plane geometry and that offered to students who had already studied some advanced secondary school

mathematics. But as far as future work was concerned we could make no distinction. The freshmen are under the jurisdiction of a separate dean and faculty whose job it is to deal with the students for one year only. We were confronted with the task of devising a one year course in mathematics of maximum benefit to approximately 600 students. Mathematics is not required. It is elected usually by about two-thirds of each freshman class. For many of these students we know that this course will complete their education in mathematics. An appreciable number will enter engineering and scientific courses where further study of mathematics is required. Some others will continue the study from various motives. All must take the same course in the Freshman Year.

The problem before us was approached with open minds. We asked ourselves the question, "In the whole range of collegiate mathematics what is most important in the modern world?" The answer is obviously the calculus. The next question was this. "Without any background of analytic geometry, how much calculus can be taught in one year?" It was in answer to this question that, from a pedagogical point of view, I feel that we made most progress. We found by limiting our aim to the exposition of the ideas underlying analytic geometry and the calculus that we could cover almost the entire subject. The standard text gave no applications of the derivative until the student had learned to differentiate every kind of expression that he was likely to meet in any undergraduate work. Two-thirds of the time allotted to integral calculus was spent in acquiring the technique of integration before any use was made of the idea. We decided that technique should be a minor consideration in our freshman course and, until the last month, we use only algebraic functions. The technique of differentiation and integration consumes only a few lessons, leaving ample time for the ideas underlying rates, maxima and minima, curve tracing in rectangular coordinates, and a wide variety of summation problems.

There can be little doubt that the students who spend only one college year in the study of mathematics have not suffered by the change. On the other hand there is distinct evidence that they have gained in a grasp of ideas unclouded by a mass of technical operations which they would never use. But the most gratifying result of the experiment is that the students who continue the study of mathematics have also gained. For most of them the work in sophomore year is a continuation of analytic geometry and calculus. They get very few

new ideas but they acquire more technique. Ostensibly there is no review. Actually everything is reviewed. The emphasis is placed on the logarithmic, exponential, and trigonometric functions and this involves necessarily a knowledge of all the preceding work with algebraic functions. Polar coordinates and analytic geometry of three dimensions are introduced, involving partial differentiation and multiple integration. The fundamental ideas are not new but the setting provides sufficient novelty and the work does not drag. More or less unconsciously the underlying principles become an integral part of the students' mental equipment. The total content of the two years' course for engineering students has not changed. But the individual topics have been broken up and rearranged. There is no idea in the first year which is not met again in the second, and yet there is no loss of time. We have found the new course to be a distinct gain over the old one.

Conclusion. The experience of the last ten years in teaching mathematics to college students is the basis for my firm belief in the advantage of applying the same general plan to secondary school work. As a parallel case we consider the course in algebra which may extend for one year in the case of some pupils and for two years in the case of others. What should be the primary aim of the first year course? I should say not technique but rather an acquaintance with ideas. Excessive drill in manipulation is of no value to the one year students and I believe that any time spent in the exposition of ideas worth while for the one year students is equally well spent for the two year pupils. The manipulative skill which the latter should have can be stressed in the second year.

I am now ready to give an answer to the question as to when the function concept should be most prominent. Emphatically the answer is the first year. We all know how this can be done if a little time is given to it. If anything in my suggestion is new, it is the thought that time should be found even at the expense of drill in manipulation.

III. Extension of the Function Concept

One effect of our democratic ideals in education has been too much of the lockstep drill. We have been too long hampered by the idea that students, free and equal by birth, should be given the same opportunities and held to the same requirements. For a long time the fact that some students did prove to be better than others had little effect on educational methods. These students often made their own

opportunities and forged ahead in spite of the system. In many cases they were encouraged and helped by their teachers but too often the routine teacher and unusual pupil have reminded us of the mother duck and her ugly duckling. Even in exaggerated cases the teachers are not always to blame. As long as our schools and colleges are filled with poor and indifferent students, these pupils will keep their teachers, for the most part, fully occupied. They attract the attention and provide the pressing problems. But more and more we are giving our attention to the problems of the better students. Recent curricular changes at Yale and many other places are looking toward greater opportunities for students who can and will take advantage of them.

It seems to me that the recommendations concerning the function concept in the report of the National Committee can be fairly well covered in the work of first year algebra. For the majority of pupils it can be merely an incidental matter in later work, not to be omitted altogether but not to be stressed at the expense of the particular topic involved. For the better pupils, those who do the work well and are likely to go on with the study of mathematics in college, I feel that even more emphasis on the function concept is valuable and in this connection I should like to give one or two concrete illustrations of my thought.

The Negative Answer. One particular idea has developed from an incident which caught my attention while we were reading the algebra papers for the College Board recently. One of the problems led to a quadratic equation which had a negative answer. One of the candidates solved the problem correctly and then started to "explain the negative answer." His attempted explanation meant nothing and it was obvious that he was trying to quote from memory some statement that had been supplied, probably by a misguided teacher, to meet any such emergency on an examination. The explanation of the negative answer is not now fashionable and probably never was very popular. Occasionally a textbook asks the pupil to explain the negative answer but rarely gives any hint as to what kind of explanation is expected. Probably an acceptable answer is somewhat as follows:

Consider the problem: *A broker bought some stock for $1000. When the price had advanced $20 per share, he sold all but one share, receiving $1080. How many shares did he buy?*

If we let x equal the number of shares bought, the problem leads to a quadratic equation giving the answers 10 and —5. For the positive answer the broker bought 10 shares at $100 each, paying $1000,

and sold 9 shares at \$120 each, receiving \$1080. The explanation of the negative answer is that it belongs to another problem which might have the following statement.

A broker sold some stock for \$1000. When the price had dropped \$20 per share he bought one more share than he had sold, paying \$1080. How many shares did he sell?

He sold 5 shares at \$200 each, receiving \$1000. He bought 6 shares at \$180 each, paying \$1080. The explanation is that negative means opposite. The opposite of buying is selling. The opposite of advance in price is drop in price. The opposite of selling one less share is buying one more share.

This is all very good and an excellent exercise for the better students. It recalls the fundamental idea of the negative sign. It brings out a point which is of importance in advanced mathematics, namely that the x of the original problem is defined for positive values only. The negative value of x has no meaning in the original problem. There is, of course, no use made of the function concept in this explanation.

Consider now a second problem: *Two airplanes race 1100 miles. The winning plane wins by one hour by going 10 miles an hour faster than the other plane. What was the winner's time and speed?*

If we let x equal the number of hours of the winner the problem leads to a quadratic equation having the roots 10 and —11. For the positive answer we find that the winner covers 1100 miles at a speed of 110 miles an hour, taking 10 hours, and that the loser covers 1100 miles at a speed of 100 miles an hour, taking 11 hours. For the negative answer the problem should read as follows:

Two airplanes race 1100 miles. The losing plane loses by one hour by going 10 miles an hour slower than the other plane. What was the loser's time and speed?

Negative means opposite. When two planes are involved the opposite of one plane is the other.

Another Problem. Before considering any function concepts connected with this problem let us discuss another one.

How long will it take a ball to reach the ground if it is dropped from the top of a building 144 feet high?

We assume that no air resistance is involved and that we know the equation of a falling body, $s = 16t^2$. Setting $s = 144$, we find $t = \pm 3$. The important point here is that, under the conditions of the problem, the equation $s = 16t^2$ is valid for the brief interval of 3 seconds only. This equation has nothing to do with what happened be-

fore the ball was dropped and nothing to say about where it went after it struck. The ball might have been rubber and bounced back; it might have been made of putty and flattened itself on a cement walk; it might have been a lead ball and might have fallen into a tank of water, continuing its downward motion. But that is another story. In higher mathematics we should say that if we count time from the instant the ball is dropped the function is defined for the interval $0 \leqq t \leqq 3$. In speaking of functions in higher mathematics, we always have in mind, explicitly or implicitly, an interval for which the function is defined.

If the function $s = 16t^2$ is defined for some other interval it belongs to another problem. This may be worth investigating. Suppose we allow negative values of t to be used. We can make a table of values and interpret the results.

t	-3	-2	-1	0	1	2	3
s	144	64	16	0	16	64	144

If the function is defined for the interval $-3 \leqq t \leqq 3$ it would mean that 3 seconds before the ball was at the top of the building it was 144 feet below the top. A study of the function shows the ball leaving the ground, rising 80 feet during one second, 48 feet during another second, 16 feet during another second, and then falling as before. Here we have an interpretation of the negative answer as belonging to a problem in which the ball is thrown upward from the ground to a point on a level with the top of a building 144 feet high.

Why do negative values of t come into this problem? Simply because of the instant from which we measure time. They can be avoided in the second problem by shifting the origin of time. How is this accomplished? If we measure from the instant when the ball is thrown from the ground and denote this time by T then obviously any value of T will be 3 larger than the corresponding value of t, so that $t = T - 3$. The corresponding function would be $s = 16(T-3)^2$, defined for $0 \leqq T \leqq 6$.

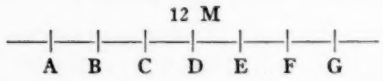
The motion of a ball thrown from the ground to a height of 144 feet and falling back to the ground is represented then by either of the two functions

$s = 16t^2$, defined for $-3 \leqq t \leqq 3$, or $s = 16(T-3)^2$, defined for $0 \leqq T \leqq 6$.

There is a point to both representations. Suppose we actually measure the time with a stop watch. We start the watch when the ball

starts, that is, we make $T = 0$ when $s = 144$. We observe that $s = 64$ when $T = 1$, $s = 16$ when $T = 2$, $s = 0$ when $T = 3$, etc. This corresponds to the second function above.

Suppose we actually measure the time with a chronograph. An idealized form of the record might look like this.



The mark A on the record was made when the ball started, the mark B when it reached a height 64 feet below the top of the building, etc. It happens that D coincides with the printed line on the record indicating 12 o'clock noon, E coincides with the printed line indicating one second after noon, C with the printed line indicating one second before noon, etc. This record suggests the first function above.

Now it was quite natural when we thought of the ball being dropped from the top of the building to measure the distance fallen *downward.* When we think of throwing the ball upward we naturally think of measuring its height from the ground upward. Here we change not only our origin but also our direction. To derive a formula for this new situation we let x denote the number of feet above the ground and note that $s = 144 - x$. The second formula above becomes

$$144 - x = 16(T - 3)^2,$$

from which we get

$$x = 16(6T - T^2), \text{ defined for } 0 \leqq T \leqq 6.$$

Variations and extensions of this problem will occur to anyone. For example we can think of throwing the ball upward or downward from the top of the building with corresponding effects produced by extending the interval of definition, or we can introduce quite different settings. Enough has already been said to indicate a possible extension of the function concept of the National Committee for the benefit of the better students in the latter part of the course. If the teacher happens to have an unusually good group he can hardly resist the temptation to go a step further and talk about average velocity and finally instantaneous velocity. But even if this is not done several important concepts are involved. The fundamental one is that of ***representing completely the history of some event by an equation showing relations between changing quantities, relations which hold for a limited time.*** There is the idea of doing this in different ways, of shifting

the origin, of changing direction, etc., all extremely valuable from the point of view indicated by the National Committee and especially so in anticipation of further study.

The Airplane Problem. Let us return to the airplane race problem. We said that the winning plane averages 110 miles an hour for 10 hours. Let us stimulate our imagination and visualize the race in detail. Let us suppose that the planes start from an airport in Colorado and fly directly westward to an airport in California. From trial flights we have estimated correctly that plane W (the winner) will average 110 miles an hour and that plane L (the loser) will average 100 miles an hour. We calculate when each plane will pass certain observation points on the route. We do this by the relations

$$(1) \qquad d_W = 100t, \qquad d_L = 110t.$$

Here are two functions. Our prognostications prove to be correct. Plane W reaches its destination when $t = 10$ and the plane L reaches the destination when $t = 11$, a whole hour later. In mathematical language the function d_w is defined for values of t such that $0 \leqq t \leqq 10$ and d_L is defined for $0 \leqq t \leqq 11$.

These numbers are convenient if the race starts at 12 o'clock because they then correspond to clock time. Suppose the start is delayed until one o'clock. In equations (1) t means hours after the race started. If T means clock time we must write the formulas as follows:

$d_W = 110(T - 1)$, defined for $1 \leqq T \leqq 11$,
$d_L = 100(T - 1)$, defined for $1 \leqq T \leqq 12$.

Suppose the race was not delayed and return to equations (1). Can any meaning be attached to these equations for negative values of t? We can make tables of values and plot the points. We find that when $t = -10$, $d_W = -1100$, and when $t = -11$, $d_L = -1100$. This corresponds to the 1100 mile race ending in Colorado and beginning 1100 miles further east, say in Ohio. Plane W gives plane L a start of one hour. The handicap has been so accurately made that the race is a tie.

Let us go to New England, 1100 miles further east, to start the race and suppose plane W gives plane L a start of 2 hours. This would correspond to $t = -20$ in $d_W = 110t$ and $t = -22$ in $d_L = 100t$. Plane W reaches the airport in Ohio when $t = -10$. Plane L reaches the airport in Ohio when $t = -11$. If the race is for only 1100 miles than plane L wins. It gets there first and at the time $t =$

—11, which was the negative answer in the original problem. If the race is for 2200 miles it is a tie. Both planes land in Colorado at the same instant. If the race is for 3300 miles then plane W wins. It lands in California 10 hours after passing over the airport in Colorado. Plane L lands in California 11 hours after passing over the airport in Colorado.

IV. Value of the Concept

Although the preceding discussion grew out of an attempt to give a meaning to a negative answer I am not advocating this type of explanation as having great value for that particular purpose. Its value, in my opinion, lies in the fact that it requires and encourages a certain type of thinking. It prepares for that kind of thought which causes most trouble to students throughout much of their later work in mathematics. As an illustration consider the following problem:

A grain dealer has in an elevator 20,000 bushels of new wheat for which he paid $2 per bushel. There is a shrinkage of 400 bushels per month, a fixed storage and interest charge of $400 per month, and a promised advance in price of 10 cents per bushel per month. If these conditions prevail indefinitely, when should the dealer sell the wheat in order to obtain the maximum return on his investment?

Solution: Suppose the wheat is sold after x months. Then $20{,}000 - 400x =$ number of bushels sold.

$2 + 0.1x =$ price per bushel, in dollars.

$(20{,}000 - 400x)(2 + 0.1x) =$

$40{,}000 + 1200x - 40x^2 =$ number of dollars received.

$40{,}000 + 400x =$ number of dollars spent.

Therefore (1) $P = 800x - 40x^2 =$ number of dollars profit.

Differentiating, we get

$$\frac{dP}{dx} = 800 - 80x.$$

Setting the derivative equal to zero, we find $x = 10$. Therefore the dealer will make his greatest profit if he holds the wheat exactly 10 months.

The solution of this problem involves two distinct steps. The first part consists in finding equation (1), that is in setting up the function, and makes use only of elementary algebra. The second part consists in differentiating the function, setting the derivative equal to zero, and solving for x. The second part of the solution presents no difficulty but, for many students, the first part is troublesome when met for the first time. It is troublesome because no set of rules can

be followed and the student must depend upon his ability to think in a particular way. He must be familiar with the function concept and the value of acquiring this familiarity as early as possible is obvious.

The preceding solution is typical of most applications of mathematics. The first part involves a statement of the problem in the form of equations and usually makes use of the function concept. The second part involves manipulation which may be simple or difficult. Both the function concept and the technique are important for the applied mathematician and technique alone is worthless.

The educated layman makes little, if any, use of mathematical manipulation. The values to him of a study of the subject lie chiefly in the development of certain habits of thinking and in clear cut methods of presenting an argument. Both of these values may be derived from any secondary school course in mathematics but the presentation of an argument is stressed most in demonstrative geometry while analytical methods of thinking may be stressed most in algebra. Skill and accuracy in manipulation are worth cultivating but have less permanent value for the average man and woman than the study of relationships included in a broad interpretation of the function concept.

For the great majority of secondary school pupils the use of the function concept, as explained by the National Committee, is all that can be attempted. Unfortunately there are good teachers who think that it represents more than can be accomplished. But for the better teachers who have the privilege of dealing with the better pupils I believe that a gradual extension from the first rough ideas of relationship and dependence to a more precise formulation under varying conditions is of great value regardless of the future career of the pupil.

ON TO MINNEAPOLIS!

February 24 and 25, 1933
National Council Headquarters
Hotel Nicollet

The program of the Fourteenth Annual Meeting of the National Council of Teachers of Mathematics will be found on page 54 of this issue.

What Do We Owe to the Brighter Pupil?

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IT IS the ideal of democracy that we should afford each pupil an opportunity to grow intellectually up to the limit of his ability. In striving to attain this ideal, educators have placed great emphasis upon the problem of the poorer pupil. Scientific studies of such pupils have been made; subject matter has been diluted to fit their abilities, and pedagogic skill has been judged by its success in saving these pupils from the ranks of failures. Probably there is much to be said for this attention to the slow learner and doubtless it has been one of the successes of modern education that these pupils have been studied and educated up to their abilities, for in raising the level of the lowest we may have raised the level of the whole mass. If, however, we have exaggerated the importance of training the poorer pupil and have placed undue emphasis upon our duty to him to the consequent neglect of the more able pupil, our loss may have been greater than our gain, for we may have substituted for the two extremes in accomplishment the one dead level of mediocrity.

There can be no doubt that this democratic policy in modern education has led to the serious neglect of the brighter pupils. Such pupils have so easily attained the goals set for the dull and mediocre, they have seemed so competent to educate themselves with little assistance from us, that we have not recognized their education as a problem but have allowed them to fall short of their superior abilities. This neglect has resulted in a serious loss to our sciences, our industries, our culture and our civic life, for all of the great advances in civilization are due to individuals of superior mentality. Early recognition of unusual ability, followed by stimulation of the highly endowed to their best efforts, would bring excellent returns to government and to society.

As early as 400 B.C. Plato, in his New Republic, speculated upon methods of identifying the intellectually gifted in order that they might be educated as leaders in his Utopian state, and he concluded that such leaders must be chosen while they were still young. In the

twenty-three hundred years that have elapsed since Plato reached this conclusion, little advance has been made in discovering or training the intellectually gifted, although European nations have been less remiss in this respect than the United States. England places great emphasis upon the education of the highly endowed, and Germany is said to give the greatest official recognition of any nation to special education on the basis of mental ability. It is unfortunate that the United States has neglected this important phase of education and it might well be one of our serious educational problems to discover the potential leaders in our schools and to stimulate them to their highest development.

Mathematics is peculiarly adapted to the discovery of unusual ability as well as to the development of such ability. Skill and ingenuity in mathematics have long been synonymous with a high grade of mentality; because of the challenge which mathematics offers to superior minds, many of the ablest pupils in a school are found in the elective courses in mathematics, attracted by the magnetic power of the subject. Moreover, since it is one of the chief aims in the teaching of mathematics to train in habits of independent thinking, we teachers of mathematics have much to contribute in the training of superior minds. It is probably in this training that we can make our most valuable contribution to society. Changing conditions demand careful analyses of new situations and solutions for new difficulties. It is probably trite to say that we are living in a changing world for it is said that Adam remarked to Eve in the Garden of Eden;—"My dear, we are living in a period of transition." Doubtless every period seems to those who are living in it a period of flux and stress; life becomes increasingly complex; but with all of the change and complexity there will always be a demand for straight and independent thinking. George Bernard Shaw once addressed a company as follows: "I suppose that you seldom think. Few people think more than two or three times a year. I have made an international reputation for myself by thinking once or twice a week." Thinking may not be as rare as Mr. Shaw would have us believe but it can scarcely be classified as one of our popular indoor sports. In Chicago we willingly admit that more straight thinking on the part of our leaders would be of great value in our local government; we even have the temerity to think that our national government might be benefited by more logical thinking in the higher circles and the upper spheres in Washington. Whatever mathematics can contribute to a habit of critical-

mindedness, of analyzing a situation, and of thinking it through to a logical conclusion, will be of great value to society.

From time to time the psychologists rise up to dampen our enthusiasm by telling us that this training in thinking is a delusion and a snare. It is somewhat difficult to know just where they will stand on this question at any given period, and I do not pretend to say whether or not this is the open season for transfer of training. At any rate, when the conflict is on, there are always Galileos in the mathematical world who cry: "It does transfer for all that!" Abraham Lincoln, who recognized clearly the gaps in his own early education, studied geometry during his presidency because he was convinced of the power of concentration and of original thought thus developed. As teachers of mathematics, our observation tells us that something does happen to the pupil who learns to reason logically; whether it be a habit or an attitude, it seems worth while, and none the less worth while because the psychologist has failed to measure it. Even the psychologist occasionally throws us a crumb of encouragement. Professor Judd says:—"The individual who has had experience with mathematics is no longer capable of returning to the level of loose, inexact thinking which characterized his earlier methods of viewing the world. General ideas do not arise in the untrained mind but are the product of constructive thinking."

In order that the superior pupils may have the full advantage of this training, we owe it to them to discover them early. A popular method of making this discovery and a method that is very easy to administer, is the intelligence test. While these tests have often failed to classify accurately, they have more than justified themselves in the focusing of the attention of educators upon the existence of wide individual differences, for while educators have always recognized these differences to some extent and have made allowances for them in planning instruction, the great emphasis in modern education upon suiting instruction to individual needs, is due to a large extent to the epidemic of intelligence tests which has swept the country, bringing these individual differences into prominence. Perhaps it is unimportant that they have failed to measure intelligence. There is something very elusive about intelligence which seems to defy measurement and to resist the methods of the stop-watch and the mechanical marking key. It is a comparatively easy task to measure the ability of a pupil to spell or to punctuate, but much more difficult to measure his literary appreciation; it is relatively easy to measure his skill in mul-

tiplication or division but much more difficult to measure his ability to attack thought problems in algebra or originals in geometry. Yet the brand of intelligence upon which we place the highest value is the very one which can be least easily measured. Since the emphasis in the grades is upon skills, the intelligence test is a much better method here for choosing pupils of exceptional ability and of determining the instruction which should be offered them, than in the high school where the more specialized subjects necessarily place the emphasis upon the reflective processes. It is unfortunate that young and inexperienced teachers place so much reliance upon these tests in estimating abilities of pupils. A pupil may lack the ability to respond quickly to the demands of such tests and yet possess the more desirable qualities of thoughtful discrimination and power of concentration which would make him a superior pupil in mathematics and a leader of unusual power.

It is fortunate for the cause of real education that we cannot rely blindly upon mechanical methods of discovering the superior pupil. At their best these tests can be used only as an aid, and the mathematics teacher still has the duty and the privilege of devising his own methods for discovering pupils who have a real flair for the subject as well as those who can attain the standard of work of which they are capable only when encouraged and stimulated. An alert teacher discovers in the course of his regular class-room work not only existing abilities but those latent tendencies which so frequently elude the intelligence tester but which are a great factor in ultimate success. An inspiring geometry teacher, in the very process of teaching congruent triangles, in explaining the methods of proof, and in leading pupils to analyze the originals to discover a successful method of attack, can discover by the re-actions of his class, by the class room discussion, by the quality of the home-work, and by the mastery shown in the tests administered, more about the abilities and the needs of his group than by any standard test. He may not be able to present his results in the form of statistics, charts or graphs; he may not be able to discuss them in terms of association shifting, ratiocination or multiple response; he may not have graphed the attention curve of each pupil to discover whether it resembles a linear function in one variable or the spiral of Archimedes, but he has obtained reliable information in a human way; he has substituted for mechanical means a method which allows for the personal equation, for inspiration, for human contacts; and in so doing he has helped to save education

from the dominion of the machine age, to prevent mass instruction from crushing out all individuality, and has left us opportunity for creative work in the teaching profession. Mechanical methods may be necessary in some phases of our work with the poorer pupils who lack originality or the ability to cultivate it; in the case of the brighter pupils mechanical methods should be reduced to a minimum if we hope to encourage independent thinking.

While much has been said about the important rôle which interest plays in awakening and stimulating bright pupils, little has been said about the necessity for hard work for really worth while accomplishment. The theorem that the arousing of interest will result in work, has had many decades in which to justify itself, and yet its frequent failure is obvious. Many teachers of mathematics make excellent use of the fascinating material and the interesting practical applications of the subject to arouse enthusiasm in the class room only to find that the enthusiasm evaporates and leaves little in the way of results. This is exactly what happens under those teaching methods which require no home work and no "follow up" on the part of the pupil. The fires of enthusiasm have been kindled and have burned temporarily but have died because of the lack of fuel which must be furnished in serious work. More emphasis should be placed upon the efficacy of work in securing satisfactory results from our bright pupils. It is not necessary to take the position of our Chicago philosopher, Mr. Dooley, who said: "It doesn't matter what you teach a child as long as he doesn't like it," but we should recognize the fact that work arouses interest fully as often as interest arouses a willingness and a desire to work. It is a mortifying experience to learn that a pupil who seemed dull in mathematics has later startled the world by his ingenuity, and yet many such pupils are not discovered or developed because the tasks set for them did not stimulate their mental powers. In school their brains became dull through inactivity and they fell into habits of indolence, conceit, and satisfaction with mediocre results. More difficult tasks might have precipitated the ability which lay in solution, but the salvation of such pupils through a régime of soft pedagogy will never be possible.

In the grades, when brighter pupils have been discovered, they are often accelerated in order to keep them working up to their ability. This may be wise in the elementary school where the emphasis is placed upon skills rather than upon reflective thinking, but in high school where there is abundant opportunity to enrich the courses in

mathematics, it is a great injustice to the able pupils to allow them to do a Marathon rapidly over the field with the emphasis upon the rate at which they cover the ground. Their course should be broadened and deepened rather than accelerated. Allowing the brighter pupil to enroll for more than the usual number of subjects is often equally injurious to his best educational interests. Except where economic necessity or other exceptional conditions make this accelerated program desirable, the superior pupil should be encouraged to improve the quality of his work rather than the quantity or the rate of doing it; his superior mentality should not be dissipated in a speed test but should be challenged by a more difficult type of work; he should be made to feel that his superior talents demand superior accomplishments.

A method which is often recommended as suited to the needs of pupils of different degrees of ability, and as conducive to securing the best results from superior pupils, is the individual or laboratory method which allows each pupil to proceed at his own rate. Under this method as it is usually administered, the superior pupil loses much that is valuable. He is tempted and even encouraged to proceed so rapidly that he misses much of the mathematical scenery of the territory through which he is passing. The different rates of the individuals in the group preclude the possibility of the daily group discussion and instruction which are such a valuable feature of other methods, and this is an irreparable loss to the superior pupil. He may have learned the method of completing the square in a quadratic equation, he may have done a difficult set of such problems perfectly, and still lack a real understanding of the method and an appreciation of the process. It is one of the important functions of the teacher to halt the pupil who knows how to do the work, and to discuss with him the meaning and the significance of the method, to emphasize the reasons for the process rather than the skill in performing it. No pupil, if left to his own devices, will emphasize this important side of mathematical training sufficiently. Nor can the teacher hope to discharge this duty conscientiously by working separately with each individual. The division of a forty-five minute class period into thirty or more equal parts so that each pupil may have his fair share of the assistance and the inspiration of the teacher, gives a quotient so small that the value is almost negligible. In the effort to help each individual, the teacher quite naturally gives the greater part of his attention to the weaker ones in the group, who are always in greater

need of assistance, and neglects the brighter ones who seem more self-reliant. In some cases the brighter pupils are pressed into service in teaching the weaker ones. This practice is unjust to the able pupil who profits little if any from the experience, and is of little value to the weaker pupil who requires expert guidance if he is to understand not only the "how" of a process but the indispensable "why." Some of us pride ourselves that we have a real gift for teaching, or at least that we have developed skill and technique in our profession through our experience in dealing with pupils. We can scarcely expect to find this natural gift for teaching in the brighter pupil; anything that he accomplishes in the rôle of pedagogue will be due to careful supervision and training on the part of the teacher, and this training in the rôle of assistant is not our debt to the superior pupil. The laboratory or individual method may help in the problem of the weaker pupil, but it does not solve the problem of the brighter; he obtains little in the way of helpful suggestion from the teacher; in proceeding rapidly by himself he misses a real understanding of the mathematical principles involved in his work; he loses the valuable interchange of ideas and methods which well conducted group discussion would provide, and the method does not inspire him to the full development of his abilities.

The method of classification into groups based upon ability is frequently relied upon to solve the problem of fair play to the superior pupil although from the point of view of the best interests of the pupils, the verdict is not unanimously in favor of such classification. Some of our progressive educators deprecate this segregation because they feel that different types of pupils can and should aid one another, that the association of different types in one class leads to a mutual understanding and sympathy, that this association has its parallel in life where citizens of varying degrees of ability must work together in clubs and other organizations, and that for these reasons a similar grouping in school without reference to ability is a fine training for life. From an administrative point of view classification also presents many difficulties. In small high schools the number of classes makes such a grouping impossible; in larger schools it is difficult to arrange satisfactorily for re-adjustments after the original grouping is made, and yet it is generally admitted that such re-adjustments are necessary to the success of the scheme since no plan for classification has been found—intelligence or prognostic tests, previous records in

the subject, general school records, or a combination of all of these—which is sufficiently reliable to obviate a later shifting from one group to another. Also since classification, to be successful, should continue through all of the required work in the subject, the problem becomes increasingly difficult and it is not unusual to find that a grouping which has begun in the first year has completely broken down in the second year, making it again necessary to teach pupils in heterogeneous groups. At best classification merely simplifies the problem of instructing the brighter pupil since individual differences continue to exist in classified groups.

Since there are serious objections from an educational point of view to the individual or laboratory method, and serious objections from both the educational and the administrative point of view to the method of classification, our problem of discharging our duty to the brighter pupil must in most cases be solved in the heterogeneous group. It is possible, fortunately, to plan graded assignments for such groups, to make the discussion period a profitable one for pupils of various levels of ability, and to devise graded tests which will separate the group according to mental equipment, diagnose individual difficulties, and offer opportunities to the brighter pupil to show the high standard of accomplishment which we demand of him.

Classification of pupils into groups based upon ability is reported to have greatly lessened the mortality among the weaker pupils. This result is accomplished by lowering the requirements made of such pupils and obviously the same result can be accomplished in a heterogeneous group by the same means. It is also claimed for classification that the segregation of the superior pupils in one group allows them to reach a higher level of attainment; again the same result can be accomplished in the mixed group by making the assignment for the brighter pupil more difficult in character. It is comparatively easy in any course in mathematics to list the minimum requirements for the passing pupil, the problems which are desirable but not absolutely necessary for those of average ability, and finally the work which requires the highest type of mathematical thinking and which should be attempted only by the superior pupils. If daily assignments are made on this basis each pupil has a task set for him which corresponds to his mental powers; the weaker pupil is not discouraged by attempting the impossible, and the ability of the brighter pupil is not dissipated in performing tasks which are too easy for him. This more

difficult assignment for those of superior mentality combines the advantages claimed for classification with the advantages which come from association between individuals of varying ability.

Graded assignments require sources for material from which a teacher may select problems for superior pupils. Modern text-books furnish a wealth of simple material for the weak pupil; practically every revision of a text in the last ten years has been a revision downward with the result that there is no difficulty in organizing a diluted course for the slow learner, but few texts offer a strengthening diet for those of superior mentality. This condition is another illustration of the emphasis which we have placed upon our duty to the weak and of the injustice which we have done to the brighter pupil. Lists of starred problems are too rare. When teachers are forced to compile their own lists and to contrive methods of getting these lists into the hands of the pupils, the difficulties become so great that it is small wonder that the needs of the superior pupil are ignored. Extensive supplementary lists of interesting and difficult problems, catalogued as to subject and readily accessible to pupils, would do much to help us discharge our duty to the brighter pupil.

When graded assignments have been made on the same topic, the difference being in the depth of penetration into that topic, the entire group can profit by a general discussion period. When the teacher of a heterogeneous group allows this discussion to remain upon the level of accomplishment of the poorer pupils of the group, he not only sacrifices the brighter pupil but he labors under a mistaken idea of the value of this method to the weaker pupil. The weaker pupil can make his contribution to the discussion by presenting the minimum essentials which he has mastered, but he can also profit by the contributions of the more able just as an amateur at golf or bridge can learn to improve his game by watching experts. The brighter pupil may profit by focusing his attention upon the fundamental principle for the day, which he may have slighted to some extent in his enthusiasm for dashing on into greener fields, but he loses interest if the discussion stops at that point. He should be encouraged to present other proofs of the theorem and applications of the principle to increasingly difficult exercises. The natural desire to talk things over, to exchange ideas, to argue, to shine before one's mates, can be utilized by a skillful teacher to bring about excellent results. The association of weak and strong in the same discussion can bring about

a spirit of co-operation and team work, a respect for the earnest effort of the weaker and an admiration for the achievement of the more able that is a counterpart of, and a training for, the experiences of life.

Such discussions cannot be staged unless home work has been assigned and prepared, for just as water adds greatly to the attractiveness of a river, so ideas add to the value of a discussion. In groups where there has been no struggle bringing about an attitude of wanting to know, no victory bringing about an attitude of wanting to tell, no rivalry stimulating the brighter pupil to do his best, there can be no valuable discussion. Setting the stage for such a discussion by making a careful assignment of home work, planning the discussion so that it becomes a valuable teaching period, and leaving the discussion at the strategic point where enough interest has been aroused and enough help has been given to challenge the group to their best efforts for the next day, is one of the accomplishments which makes teaching a fine art.

Conference periods in or outside of school hours have been used rather extensively as an aid to the weaker pupil, and where these periods have been offered as a means of assistance rather than as a punishment, the method has been successful in locating and remedying difficulties. It is unfortunate and unjust that the weak pupil should have a monopoly on this scheme. Teachers who have experimented with arranging conference periods with the brighter pupils are convinced of the merits of the plan. Brighter pupils are particularly appreciative of any special interest on the part of the teacher in their progress. The criticisms or commendations on a piece of difficult work, the suggestions for improvement or for a different plan of attack, are a stimulus and a spur to the highly endowed which should not be overlooked if we hope to get the best results from these pupils.

Graded tests throughout the semester and at the end of the course stimulate the brighter pupil to his best effort. If it is unfair and discouraging to ask questions of the weaker pupils which are beyond the range of their ability, it is equally unjust to the brighter pupils to demand of them nothing which taxes their mental powers. Hurdle tests used as teaching, diagnostic, or accomplishment tests stimulate the better pupils to a high standard of work, and the scores, whether kept individually or displayed publicly, are an incentive to improvement. There is nothing which gives the better pupil a greater feeling

of satisfaction at the end of a course than the ability to pass with credit a power examination which gives him an opportunity to apply his knowledge to new situations.

Whether or not we believe in rivalry as a spur to superior work, we are compelled to admit that it exists and since that is the case it may as well be utilized for legitimate ends. Just as we attempt to cultivate a spirit of good sportmanship and of generous admiration for a rival in athletic contests, so we should train pupils to admire unusual mental accomplishments. The competitive nature of the work at West Point and Annapolis has been a means of attaining and preserving a high standard of work in those schools and afford us an excellent illustration of the legitimate use of competition as a stimulus to good work.

One of the best methods of combining honor with opportunity for superior pupils in mathematics is through student mathematics clubs. Since I know so little of the ways in which they have aided other schools, I should like to tell you briefly of our pleasant and profitable experiences with these clubs at Hyde Park. We organized our first mathematics club several years ago for pupils who had done excellent work in the department and who seemed interested in doing more work than was possible in the regular classes. Pupils were eligible who had finished four semesters of mathematics with an average of "Excellent"; meetings were held every two weeks under the sponsorship of a member of the mathematics faculty, and programs were planned by a committee of club members in consultation with the sponsor. From the beginning the club was a great success. It brought together from the various classes pupils who had real interest, real enthusiasm, and real ability in mathematics, and membership in the club was regarded as a great honor as well as an enviable opportunity. After a few years the number of eligible applicants became so great that we decided to form two clubs—the Euclidean Club which is open to Juniors who have completed four semesters of mathematics with an average of "Excellent" and Pythagorean Club which is open to Seniors who have finished three years of mathematics with an average between "Excellent" and "Superior"—an average somewhat higher than that demanded of the Euclideans. Because of the difference in grading in various classes, there have always been a few applicants for the Pythagorean Club who seemed to have the ability, but who lacked the requirement in grades; for this reason we have been experimenting this year with a group known as the "Wranglers" who

have met with the Pythagoreans and have attempted to show the club and its sponsor that they were worthy of membership.

The organization of the clubs has been kept very simple. Each club elects a President and a Secretary-Treasurer and the president appoints such committees as he thinks necessary. The programs vary with the interests of the members. Difficult problems are proposed by members, by teachers, by business men in the community, and by former members who send back suitable problems from their college work; mathematical fallacies and recreations and the history of mathematics always prove interesting; topics not included in the regular courses are discussed, such as theoretical work in logarithms, incommensurable cases and the theory of limits, simple problems in analytics and the calculus with some of the simple applications to maxima and minima. Frequent contests are held on review topics or on recent college examinations. Often an invitation is issued to other students in mathematics to compete with the club members in these contests and the enthusiasm and interest runs high when a non-member scores in such a contest. The weekly school paper carries notice of the meetings and gives favorable mention to members who have contributed to the program and to the winners in contests. Regular attendance at the meetings and active participation in the programs entitle the members to wear the club pin and to appear in the club picture in the school annual, both of which are much coveted honors.

The enthusiasm and interest on the part of the pupils has been very gratifying. In spite of the serious purpose of these clubs, they flourish in a school which has a multiplicity of clubs, the majority of which are social in character, and where there are many conflicts in time of meeting so that it is necessary to choose between them. While it is difficult to measure the results of these organizations, some of them at least are tangible. Many of our Pythagoreans enter technical or engineering schools where their records show that they continue their excellent work in mathematics. Every year Pythagoreans are successful and rank high in College Entrance Board examinations as well as in the competitive examinations for West Point and Annapolis. Each year some of the Pythagoreans compete with candidates from other states in the prize scholarship examinations at the University of Chicago. The club is especially proud of its record in these examinations, for in the four years that the University has followed its present plan of conducting these competitions, the Pythagoreans have won all of the full scholarships offered in mathematics as well

as many partial scholarships and honorable mentions. Not the least of the results of these clubs has been the real interest in mathematics which has been developed and fostered. Our mathematics department feels that these clubs have a very pardonable pride in the successes of their members and that they have been invaluable in stimulating our better students to do a high standard of original work.

Although the problem of discharging our debt to the brighter pupil is vitally important to educational progress, we have seriously neglected it in our enthusiasm for educating the masses. Educational literature has given little attention to the needs of the brighter pupils; text-books have provided for pupils of low mentality but have ignored those of superior ability; class-room methods have been adapted to the needs of the slow or average learner. Until we teachers recognize the injustice which has been done to the upper group and the duty which we owe this group, until we plan methods for interesting the brighter pupils and for stimulating them to their best efforts, we shall fail in realizing our ideal of education and in our contribution to national progress.

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Mathematics and Social Science

By ELLA M. PRENDERGAST
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HAVING felt for some years the increasing lack of co-ordination among the departmentalized units of the junior high school, the writer unexpectedly received an impetus toward active work on this problem as related to mathematics, during the progress of a graduate course in the "Teaching of Mathematics," taken at the University of Southern California under Dr. Myrtie Collier. Through her encouragement and under her helpful supervision, some experimental investigations were started which it is hoped may have their small share in turning professional thought toward correlation of subjects in a definite and constructive manner.

A preliminary survey of the subject, and some research work revealed a number of interesting facts. Most important of these was the revelation of the increasing need of a knowledge of mathematics for the full appreciation and correct understanding of practically every other subject in the curriculum. As long ago as 1922, Touton of the University of Southern California, voiced this need when he said[1]—"The outstanding finding of the study, the great frequency of mathematical concepts in general literature, confirms the conviction of teachers of mathematics as to the importance to be attached to a full understanding of quantitative terms."

A second fact of importance to become apparent was, that while correlation is becoming more prevalent as a subject in current educational magazines and books, the field of mathematics has been very sparingly treated. Two or three noteworthy exceptions were found. First, the excellent work started by Edith Long in the high school courses at Lincoln, Nebraska. Through several years of devoted effort she brought about a change in the curriculum there which resulted in splendidly correlated courses in mathematics and physics. The movement which she initiated was aided by the epoch-making

[1] Touton, F., "Mathematical Concepts in Current Literature," *School Science and Mathematics*, 1922

speech of Moore.[2] He aroused a national feeling which resulted in excellent effort being brought to bear upon the subject throughout the country, and splendidly related courses are gradually being worked out between the sciences and mathematics. One very recent report comes from the University of Pennsylvania.[3] A mathematics course has been prepared for college freshmen entering science work, in which the particular mathematics deemed necessary is reviewed, and which must be finished as a prerequisite to the science courses. The interdependence between science and mathematics has of necessity forced itself upon those engaged in teaching these subjects, and so is bringing about the necessary correlation.

But the sciences play little part in the junior high school, as the courses there are very general and for exploratory purposes only. And it is in the junior high school that a much greater need has been entirely neglected, because it has not been so readily discernible. This is the need for correlation between mathematics and the social sciences. Just as it has now been acknowledged that geography, history, occupations, and the like, constitute an inseparable unit, so it must inevitably be seen that they are unworthy of the name of social *sciences* unless they incorporate in their work the quality of measurement which only mathematics can contribute. In a masterly article in the Sixth Year Book of the National Council of Mathematics Teachers, Professor Irving Fisher of Yale says—"Just as the multiplication table is applicable in more than one field of knowledge, so mathematics in general is peculiar to none. Sooner or later every true science tends to become mathematical." His own outstanding work in bringing a truly scientific attitude to social statistics is a brilliant example of what a combination of mathematics and social science can jointly accomplish.

If a reorganization of the curriculum can be brought about which will allow these two branches of instruction to become co-ordinated, it cannot fail to bring about a revolution in curriculum making. All subjects would sense their own need and demand a grade by grade adjustment whenever there was any opportunity for so doing. And co-ordination once begun in the junior high school would inevitably have a regenerating effect upon the senior high school.

[2] Moore, E. H., "Foundations of Mathematics," First Year Book, National Council of Teachers of Mathematics.

[3] Evans, H. B., "Co-operation in Mathematics and Science," MATHEMATICS TEACHER, January, 1932.

In casting about for an opportunity to begin some sort of actual correlation work in the classroom, the writer took whatever came to hand as determined by the classes she had at the time. A B8 class engaged in a unit on "problem solving" was the first to present itself. After a short class discussion of the question—"Do we use our mathematics in any other classes?"—an assignment was made requiring at least three outside problems from each member of the class. It was felt that this would advance the study of the unit in many ways. The ability to state a problem correctly, giving all the necessary data and stating clearly the relation between quantities, as well as making plain just what is required to be found, brings to problem solving a greater degree of understanding than can be developed in any way.

In this first assignment, problems were to be restricted to those found in other classes, and the subject and grade in which they occurred were to be indicated. As might be expected, the results did not show a wide range. They were rather startled to find that they were using any mathematics anywhere except in a mathematics class. Briefly stated, their problems consisted of the following,—finding grades in per cent on their tests in various classes; reading map distances by means of scales, and finding differences between dates in social studies; dividing circles into equal parts and getting correct proportions in art; using proportion in changing recipes in cooking; finding costs of materials in sewing; a few woodshop problems; and one or two problems from science.

These problems were discussed in class, wording clarified in some cases, and some class exercises on solving done. The main object of the collection, however, was to come later with their incorporation in the mathematics course at the proper time and in correlation with the classes in which they occurred. The correlation of "time" in all of this work was found to be most important. Teaching board feet in a B8 mathematics class was certainly not of much use to the 7A shop teacher. Certainly little difficulty need be found in making an adjustment. Many such examples might be pointed out.

Later, the class was asked to bring in some problems from any source outside of school or otherwise. By this time they had caught the spirit of the thing, and a most remarkable set of problems resulted. A number of days were given to discussion, especially as to wording and assembling of all the facts needed. More than once it was found that the question father had so glibly propounded at son's solicitation, could not be answered without further data. English was

polished up in an attempt to put in the clear and exact language of mathematics the thought the young originator wished to bring to the class. This collection of the problems, when collected and compiled, will furnish work for future classes. Perhaps this compilation itself will form a class project.

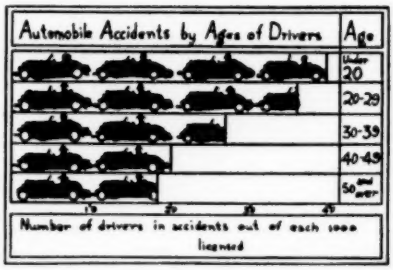
This project is to be continued through the coming year by means of a questionnaire to be placed in each teacher's box asking what mathematics they find useful and necessary in their classes, and for the statement of actual problems. If they have a preferred method of solving them, they will be asked to indicate it, and also to state in what grade the problems are used. It is hoped that by some slight readjustments these suggestions can be incorporated in the work of the mathematics classes in such a way as to actually correlate with the other subjects, and thus make a small start toward bringing the departments together in spirit and in fact.

A small project undertaken with another class was the correlation of the study of graphs with the study of safety, which was being given considerable time under the head of an activity meeting once a week. This project gave the opportunity for developing much that was worth while in functional thinking, and at the same time provided the chance to make a start on social science material. Outside material for graphs of all kinds was brought in and a great variety of graphs produced.

In order to lead them to something beyond what they were able to do unaided, opportune articles in the paper were suggested as suitable material as well as school safety reports obtained from the principal's office. The sad lack of ability to visualize the picture these figures contained showed at the outset how needful is its development in this type of work. Further motivation was provided when the journalism department expressed a willingness to publish a series of three or four of the best safety graphs in the school paper. In order to make them more interesting to all of the school, the pictogram idea was injected, excellent suggestions along this line being obtained from a collection of graphs recently put out by the Travelers Insurance Company in a book called "Tremendous Trifles," and highly recommended to anyone interested in this type of work.

It was felt that the project was highly successful. The three graphs finally published,* with short explanatory articles, stressed three im-

* The printer's proofs on the following page show the size of the cuts, which are available at a very nominal cost.



portant safety ideas—the high ratio of accidents among the youthful driver class as compared with older groups; the position of the school as to accident record as compared with the twenty schools of the same type; and finally an optimistic note, the gradual reduction in New Jersey of the ratio of accidents to children as compared to adults over a period of ten years, largely attributed to safety education. It must have been apparent to all that the application of the mathematical study to the safety statistics had made the results available in the best possible form, and enrichment of both subjects was the result.

Percentage of children's accidents total for New Jersey, in 1922 and 1932; a reduction of 13.5 per cent due to Safety Education.

The success of these small beginnings have proven to the writer that a rich field here lies ready for cultivation by devoted workers everywhere, looking to a common goal, the improvement of instruction and the development of the whole child. In one of the latest books on "Teaching the Social Studies" by Fancler and Crawford, this significant statement appears—"Correla-

tion, at one time an educational fad, has left a definite impress upon school practice." . . . "The tendency now is to bring the social studies into vital relationship with Manual Arts, Science, English, Art, Music." Also—"Social Arithmetic text-books have been written"! The foregoing sentence sums up the attention given to mathematics.

Even here, then, it is apparent that some one directly interested in mathematics must make the true mathematical contribution in this program of correlation. Many pages are given to the correlation with other subjects, but the measuring quality of mathematics as applied directly to social studies is entirely passed over.

In an examination of the social studies curriculum which the writer is now making, much co-ordinating material has already been unearthed. The following is typical of the chance to bring out the much neglected history of mathematics as the study of various countries progresses. Among the questions in the work book we find—"What would you like to know about the Greeks?" What about the Greeks, indeed, more typical or more necessary to an understanding of their contribution to civilization than the fact that they made mathematics a science, and that the advances in geometry in particular, with which the student is more or less familiar, have remained through the centuries. The romance and mystery of the Pythagorean Societies alone should appeal to any child if properly presented.

Then again, this topic in an outline on U. S. History was most provocative—"The Empty Treasury." What is being taught in the social studies under that topic? Why cannot the social and economic side of taxes, banking, interest, and the like be developed here in a definite way, instead of retarding the progress of classes in mathematics by excursions into this field. And in the mathematics classes graphical comparisons, and definite quantitative work, would help to make the "empty treasury" a living reality to them. The present sad condition of our nation today might have been avoided if we had had more men trained to think in clear mathematical terms concerning this very subject.

The true ideal before a mathematics class should be the development of the characteristics peculiar to this subject alone, i.e., functional thinking, with its attendant power to interpret the relation between quantities, and the power to state clearly one's premises and with logical reasoning, build conclusions which do not violate them. This is best accomplished through practical applications to problems of daily living. Their contribution to a correlated program would be a

truly scientific interpretation of the many comparative quantities which occur in social studies. Statistics, in our modern civilization, are assuming increasing importance as a means of forecasting the future from the past. We must develop mathematicians with the ability to interpret this mass of figures for the good of all.

It is to be hoped that correlation will emerge from the aspect of being an educational fad, and come soon into proper perspective. May it contribute toward the building of that ideal curriculum of the future, of which Mr. Robert H. Lane, Assistant Superintendent of Los Angeles Schools speaks, when he says in an article in the *Los Angeles School Journal*—"The curriculum, therefore, is not subject-matter but the sequence of experiences which the child undergoes. These experiences necessitate . . . an integrated school program which revolves about the whole child rather than a program which is made up of heterogeneous and unconnected experiences."

Notice to Subscribers!

We have had several letters recently from former subscribers in which two one-dollar bills are said to have been enclosed to cover one year's subscription to THE MATHEMATICS TEACHER. In many such instances we have found no money in the envelopes and in some cases evidence that someone had been tampering with the envelope. In other cases it is obvious that we have received neither the money nor the letter. We have taken the matter up with the Postal Authorities and we hope we may get some relief from this unusual situation. In the future we trust that, as far as possible, all subscriptions will be paid by check.

In any case we trust that our subscribers will bear with us for sending notices when we do not receive the subscription price. If we make a mistake, we are always very glad to correct it.—THE EDITOR

Real Life Experiences in the School

By STEPHEN J. BOTSFORD, *Assistant Headmaster*
Staten Island Academy, Staten Island, New York

DO WE WANT "real life experiences" in our schools? This problem is one that many educators are discussing and is the subject for many experiments. For some time, led by Professor Kilpatrick of Teachers' College, the proponents of this method of education appeared to have the best of the argument. However, the opposition is meeting the issue so that the contest is by no means one sided and final conclusions may never be reached. There seems to be no doubt but that "experience is the best teacher," and it all depends upon what is to be taught whether or not "life experience" should be the method used.

Attending school is the real life experience of our children and the work of the school should not be in a compartment set aside from all their other life experiences. For instance, "health" is an important factor in the lives of all children, both in and out of school and it is difficult to think of carrying on a health program on any basis other than through life experience. Hands get dirty in school just the same as outside school and are made clean in the same way. Impure air has the same effect in a school room as in a bed room and a cut finger is an "experience" whether it occurs in a work shop at school or at home.

Likewise, children have experience in reading, writing, spelling and locating places on a map (city and roads) out of school as well as in school. The above experiences are common to all children and, therefore, must be considered an important part of any school program.

Arithmetic, in nearly all its forms, enters the life of every child and for this reason, as well as others, may be developed in school by the "life experience" method. It is not the purpose of this paper to prove that the occupations requiring arithmetic, particular buying, selling, and investing, should be reproduced in school, but rather to describe a piece of work done by the pupils of the Staten Island Academy, Staten Island, New York.

During the first weeks of the school year 1929-30, the pupils of the eighth year arithmetic class began to enquire about stocks and to make statements concerning the Stock Exchange. Nearly everyone had something to contribute for it was evident that the market was being dis-

cussed at home, and, of course, it was front page news at that time. Clippings from daily reports and magazine articles, as well as pamphlets issued by corporations and brokers, began to appear. The class was an open forum in every sense of the word with a teacher who rarely would act as an authority on any business practice. Here began a search for authority on the many questions that came up for discussion. Of course, appeals were made to all the fathers. One of the fathers is a manager of a bank, two are lawyers, several are bank directors, while others are members of the Board of Directors of business corporations. Frequently, statements of these men differed and calls were made on men whose positions were such that their statements were acceptable to the members of the class.

The question of how the class could purchase a radio and rent it to the school finally arose. Some one suggested that each pupil should subscribe a certain amount with no mention of where each was to get his money. When this question was asked, it was interesting to observe the expression of "Why ask foolish questions." The answers came promptly—"Ask father for it." The instructor gave unmistakable signs of disapproval of this solution only to hear—"I'll take the money out of my savings account." However, by this time, several who had been doing some thinking were asking questions and suggesting that if this were a sound proposition they might borrow money at the bank. A note was suggested. One pupil thought that they could go to the bank, sign a note, and walk out with the money. Another pupil had a brilliant idea, and fairly exploded with the word "security." "What do you mean by security?" "Oh, it is something valuable that you give to the bank to keep as long as you have the money." Watches and diamond rings, several thought could be used.

The next day, the class as well as the instructor, were surprised when two of the pupils said: "Well, we thought it was about time somebody found out some things so we went over to the bank and asked a man at the window and he told us all about it." That was an experience that will never be forgotten, and, undoubtedly, contained the germ that was to bear fruit a little later in an important enterprise.

At the next meeting of the class, the radio investment was discussed and the question asked: "How much money are we going to make on our investment?" It was rather surprising to note how many pupils thought that enough rent should be charged the first year not only to pay for the set but to give each one in the class several dollars' profit.

Now, began a real investigation: costs, up-keep, depreciation, a fair

yield on the money invested, promissory notes, endorsements, security, and discount. Some time after this the instructor witnessed in the classroom a "business transaction" that had not been previously staged, enacted or coached in any way. Two young people walked into the "bank" and stated to an attendant that they would like to make a loan. They were informed that they would have to see the "Vice-President." When the request was made to that official, he wanted to know what they were going to do with the money. The investment was discussed and the "Vice-President" convinced of its soundness. The loan was granted, provided that some one whose financial position was assured would "go on the note." This bank official was told that the borrowers had considered this but had decided that they would offer security for the loan and "stock certificates" were presented. Market value of these "securities" and the margin of safety were discussed and, finally, the "certificates" accepted, the note drawn and the money paid.

The instructor now thought that the time had come for the question for which the work of the preceding weeks had served as an introduction. "Eighth Grade, what shall we do now?" "Oh, sir, can't we go into real business?" Success! The experiment was working. Several suggestions were made and discussed, but a "School Store" was the business that a large majority were eager to try. And so it was decided, but not until nearly everyone in the class, on his or her own initiative, had emphasized and urged that the enterprise should not in any sense be "make believe," but that it was to be undertaken on a strictly business basis.

The information gained through the previous discussions and investigation was of great value at this stage for almost immediately it was suggested that a stock company be organized. The work of organization occupied the time of several weeks for this was not textbook work and the instructor did not act as an organizer or authority and, not often, as a check.

It was understood from the very beginning that, while the enterprise was most emphatically to be a business proposition, it was also to be run so that each member in the group should have the opportunity of learning all the details of organizing and conducting a business. Consequently, after asking and receiving outside information concerning an appropriate name, "The Eighth Grade Co-operative Store" was organized with the entire class as members of the "Board of Directors." Officers were elected for the first half of the year and committees ap-

pointed. The Board met every day. What was to be the amount of the Capital Stock? "Par value" of the shares was a knotty problem. It should not cost so much that the younger pupils would not purchase it, neither should it sell for so small an amount that the certificates would not be seriously treated by the older pupils. Twenty-five cents was finally agreed upon.

The stock certificates next came up for consideration and the committee sought expert advice. Such technical expressions as "fully paid" and "non-assessable" were examined and explained to all members of the Board. The problem of producing the certificates required considerable investigation, for the committee learned that the special paper and the engraving of the certificates issued by nearly all business houses would cost entirely too much. It was decided to secure a good grade of bond paper and to use the Academy's mimeographing machine. Accordingly, a stencil was cut and the certificates printed by the members of the committee. The stock was then placed on sale and several "sales talks" were given in Assembly stating the advantages of holding shares in the "Co-operative Store" and of the unusual opportunity presented for every pupil in the Staten Island Academy to make an investment that would pay him a good dividend. These "sales talks" and the actual sale of the stock to the pupils and teachers was of considerable educational value to the entire school and was the subject for general discussion.

Another problem now presented itself: Where, in the building, was the store to be located? Various possibilities, with advantages and disadvantages were discussed. A list of available rooms was obtained and one chosen nearest the Lunch Room. The rental was agreed upon and suitable furnishings obtained from the school. During all this time, several in the group who were less interested in the organization proceedings were discussing what was to be sold in the store. Lists were prepared and some time spent in deciding on the articles to be purchased for the opening. One factor that has been constant throughout the process of establishing the business should be noticed. It is the conservatism of the group. They voted, at once, to stock only those articles that would be certain of a quick sale and to go slowly in building up their merchandise. It was surprising to have so many young people display such a degree of caution, and considerable argument was necessary to convince some of the members of the Board that certain goods be added to the stock.

Two classes of merchandise were chosen—school supplies: paper,

pencils, notebooks, etc., and, of course, things to eat: cakes, crackers, candy, and ice cream. Here, a new problem presented itself—a genuine business situation that confronts every Board of Directors—the above-mentioned commodities were already for sale by the school. Would the school give up its "franchise"? Conferences were planned and arguments prepared that must have been convincing for an agreement was reached in which the Academy consented to retire as a competitor. When the school complied with this request, the project seemed assured and the time set for the opening of the Store. The Board elected a purchasing committee which immediately displayed its business ability by arranging to take over the stock owned by the school. However, how was this to be done with little money in the treasury? The Committee solved the problem by suggesting that a promissory note be offered in payment for the goods. The note proved to be acceptable and the officers were instructed to draw it. Somewhat to the surprise of the instructor, but quite what his educational tenants should cause him to expect, the officers were not able to draft a note and be certain of its correctness. This fact should be emphasized for this particular reason. At the beginning of the year, all the members of the class were given as a textbook assignment the drafting of a promissory note. Each pupil brought to class an accurately drawn note. Some might think that the lesson was learned, but it was very evident that at that time the writing of a note was much like copying a Roman inscription from an ancient temple. It was not a part of their life experience. The maker of the note came to the instructor, who professed vast ignorance of note-making, so that it demanded a little research and a trip to the bank to be certain of legal correctness. It would be interesting, in five or ten years from now, to test the young people who were responsible for the writing of that note.

The Store was stocked and duly opened. Each member of the group took his turn in acting as clerk and making a record of the daily sales. At the next meeting of the Board of Directors, the question arose, "What shall we do with our money?" "Would it be possible to open an account in a bank?" No one knew, not even the instructor; consequently, without preliminaries, two of the pupils walked over to the bank and asked to see the Manager. The instructor often wished that he might have listened in on that interview. In a short time, the delegation appeared with bank book, check book, and deposit slips. The bank had accepted the account, the treasurer could make deposits and

draw checks. At this time, several of the members thought that some accounting would be necessary. A journal and ledger were bought. The Treasurer opened these books and has been responsible for the accuracy of every entry as well as instructing several others in proper methods of bookkeeping.

The Purchasing Committee has had many business experiences. They have met salesmen, compared prices, judged values, and determined the kind and amount of goods to buy. They have tried to create a demand for certain articles and have introduced new lines of goods.

It would not be a complete story if there were no mention of the "Articles of Incorporation." The Articles of Incorporation were of slow growth, for experience and research were the principal guides. Consequently, the usual process of first writing the Articles and then organizing was reversed.

In order that as many as possible could have executive experience, new officers were elected for the second half of the year, with the understanding that they could obtain advice from the old officers if occasion warranted it.

At the close of the school year, the Directors were very busy closing out stock, taking inventory and preparing the books for audit. The gross and net earnings were determined, as well as the amount of money to be retained as undivided surplus and, finally, the dividend to be paid the stockholders. The financial report of the year's business was prepared and discussed with an experienced accountant by the officers of the corporation. This report was presented to the Board and, after being accepted, a copy was mailed with the dividend check to each stockholder.

Real life business situations and experiences constituted the work of the year. Did the pupils learn how to conduct a business? To some degree, yes. I do not know that that was the main purpose of the project, but I do know that to the extent that each pupil experienced, he learned.

Our goal is 10,000 members in the National Council of Teachers of Mathematics by 1935! Please do your part.

The Tetrahedron and Its Circumscribed Parallelepiped

By NATHAN ALTSHILLER-COURT
University of Oklahoma

GASPAR MONGE (1746-1818), like his protector and idol, the Great Napoleon, was a man of humble origin. Time has tarnished the glamor of the victories at Marengo and Austerlitz, and future generations may grow indifferent towards the name of the successful general. However, wisdom-loving humanity will forever cherish the name of Gaspar Monge not only as the father of descriptive geometry, as the great teacher of Poncelet, Brianchon, and Dupin, as the organizer of the famous École Polytechnique, but also for his many brilliant contributions to solid geometry, which he loved—to the exclusion of plane geometry.

Among these is the relation between the tetrahedron and the parallelepiped. This relation is just as ingenious and simple, as it is fruitful. It has been totally neglected by textbook writers, especially in English.

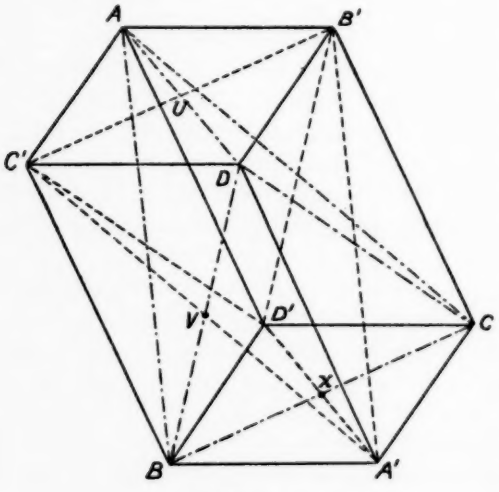


FIG. 1

1. Given a parallelepiped (P) (Fig. 1), through one of its vertices, say A, draw the three diagonals AB, AC, AD, in the three faces of (P) having the vertex A in common. The tetrahedron (T) = ABCD is said to be "inscribed" in (P), and (P) is said to be "circumscribed" about (T). In other words, the relation between (T) and (P) is characterized by the fact that a pair of opposite edges

of (T) are a pair of non-parallel diagonals of a pair of opposite faces of (P).

2. The vertices A′, B′, C′, D′, of (P) diagonally opposite to A, B, C, D, on (P) are the vertices of a second tetrahedron (T′) = A′B′C′D′, which is also inscribed in (P), for the pairs of opposite edges of (T′) are also pairs of non-parallel diagonals of pairs of opposite faces of (P).

We obtained (T) and (T′) by starting our construction with the vertex A of (P). It is easy to verify that the same two tetrahedrons would have been obtained no matter which vertex of (P) we would have started with. This may be done by actual trial, since there are only eight possible starting points for the construction considered.

3. It might be supposed that the tetrahedron (T) inscribed in (P) (1) is of some special, restricted kind. But this is not the case.

Given any tetrahedron (T) = ABCD, we may circumscribe about (T) a parallelepiped (P), and only one, by passing through the three pairs of opposite edges AB, CD; AC, BD; AD, BC, of (T) three pairs of parallel planes. For through any two skew lines* one and only one pair of parallel planes may be drawn, the two planes being perpendicular to the line of shortest distance of the two skew lines considered.

4. The two tetrahedrons (T), (T′), inscribed in (P) (2) are intimately related. It is well known that the diagonals AA′, BB′, CC′, DD′ of (P) (Fig. 1) meet in a point, say, G (not marked in the figure). It follows that the pairs of corresponding vertices A, A′; . . . of (T), (T′) are symmetrical with respect to G, hence: *The two tetrahedrons inscribed in the same parallelepiped are symmetrical with respect to the point common to the diagonals of the parallelepiped.*

Thus corresponding faces of (T), (T′), are parallel, their volumes are equal, their circumcenters are collinear with G and equidistant from G, etc.

5. Definition. The two tetrahedrons (T), (T′) (4) may be referred to as "twin" tetrahedrons.

6. The point G has a geometric significance with respect to (P), as well as with respect to the twin tetrahedrons, when these are

*Two lines in space which do not intersect and are not parallel are said to be "skew," a term which may be recommended to the attention of text-book writers.

taken together (4). But what is the rôle of G with respect to (T) alone? A first glimpse at this relation is afforded by the

THEOREM. *The three lines joining the mid-points of the three pairs of opposite edges of a tetrahedron are concurrent and bisect each other.*

Indeed, the mid-point X of the edge BC of (T) (Fig. 1) is the point of intersection of the diagonals BC, A′D′ of the face A′BD′C of (P). Similarly for the mid-point of U of AD. Now in the parallelogram BCB′C′ the line UX joining the mid-points of the two opposite sides BC, B′C′ passes through the point G common to the diagonals BB′, CC′, of the parallelogram, and UX is bisected at G. Similarly for the other pairs of opposite edges of (T). Hence the proposition.

7. The lines joining the mid-points of the pairs of opposite edges of a tetrahedron will be referred to as the "bimedians" of the tetrahedron. The term is due to the Belgian mathematician Joseph Neuberg (1840-1926).

The above proof (6) shows that the bimedians of a tetrahedron are equal and parallel to three concurrent edges of the parallelepiped circumscribed about the tetrahedron.

It may also be observed that two twin tetrahedrons have the same bimedians, and consequently their points G are identical.

8. Pursuing further the relation of the point G to the tetrahedron (T) we obtain the

THEOREM. *A diagonal of (P) (Fig. 1) meets the corresponding face of (T) in the centroid of this face.*

Indeed, in the parallelogram A′CAC′ (Figures 1 and 2) the line CV joining C to the mid-point V of A′C′ is the median of the triangle CA′C′ and therefore meets the median A′G of this triangle in the centroid S of CA′C′, and we have

$$CS:SV = 2:1.$$

But V is also the mid-point of BD, therefore S is also the centroid of the triangle CBD, and the diagonal AA′ of (P) passes through S, hence the proposition.

9. THEOREM. *The diagonals of a parallelepiped are trisected by the corresponding faces of the two inscribed tetrahedrons.*

Indeed, $A'S = 2/3A'G = 1/3A'A$ (Fig. 2).

The face *B′C′D′* trisects *AA′* near the vertex *A*. This may be shown by a reasoning similar to the above, or may be taken to be an obvious consequence of the symmetry of the two tetrahedrons (*T*), (*T′*) with respect to *G* (4).

10. Definition. The lines joining the vertices of a tetrahedron to the centroids of the respectively opposite faces will be referred to as the "medians" of the tetrahedron.

11. With this new term (10) at our disposal we are now ready to state a fundamental property of the tetrahedron, due to the Italian mathematician F. Commandino (1509-1575).

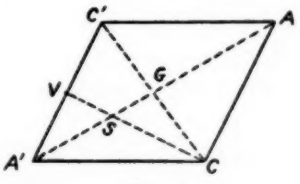


FIG. 2

THEOREM. *The four medians of a tetrahedron are concurrent.*

Indeed, we have just seen (8) that the medians of the tetrahedron lie along the diagonals of the circumscribed parallelepiped, hence these medians have the point G in common.

We may also observe that the medians of (T) are equal to two-thirds of the respective diagonals of (P) (9).

12. Definition. The point common to the medians of a tetrahedron is called the "centroid" of the tetrahedron.

We have seen already (6) that the bimedians of a tetrahedron also pass through its centroid. Two twin tetrahedrons have the same centroid, which point is the point of intersection of the diagonals of the common circumscribed parallelepiped of the twin tetrahedrons.

Consequently, the twin tetrahedron (T') of the given tetrahedron (T) is the symmetric of (T) with respect to the centroid of (T). Thus given (T), the tetrahedron (T') may be constructed, without the help of (P).

13. Commandino's theorem is analogous to the property of the triangle. This analogy is further exhibited by the

THEOREM. *The centroid of a tetrahedron divides each median in the ratio of 3:1, the longer segment being closer to the vertex.*

Indeed, $GS = 1/3GA'$, and $GA' = GA$ (Fig. 2).

14. The preceding properties of the tetrahedron may be arrived at in other ways. The ease with which they were obtained here shows the merit of the Monge idea to associate the tetrahedron with the parallelepiped. Further illustrations follow.

15. Definition. The plane passing through the mid-point of an edge of a tetrahedron and perpendicular to the opposite edge may be called a "Monge plane" of the tetrahedron. A tetrahedron thus has six Monge planes.

A plane passing through the mid-point of an edge of a tetrahedron

and perpendicular to this edge may be called the "mediator" of this edge. We know that the mediators of the six edges have a point in common, namely the circumcenter of the tetrahedron.

16. THEOREM. *The six Monge planes of a tetrahedron have a point in common.*

The mid-point X (Figures 1 and 2) of BC is also the mid-point of $A'D'$, and the edge AD of (T) is parallel to the edge $A'D'$ of (T'), hence the Monge plane (X, AD) of (T) passing through X and perpendicular to AD coincides with the mediator of $A'D'$, and therefore passes through the circumcenter of (T'). Thus the six Monge planes of (T) pass through the circumcenter of (T').

17. Definition. The point common to the Monge planes of a tetrahedron is called the "Monge point" of the tetrahedron.

Due to the symmetry of (T) and (T') with respect to G we have: *The Monge point of a tetrahedron is the symmetric of the circumcenter with respect to the centroid of the tetrahedron.*

The Monge point plays an important rôle in the study of the altitudes of the tetrahedron.

18. The two tetrahedrons $A - BCD$ and $A' - BCD$ (Fig. 1) have the same base BCD, hence their volumes are to each other as their altitudes issued from A and A' respectively. Now these altitudes are corresponding legs of two similar right triangles which have for hypotenuses the segments into which the face BCD divides the diagonal AA' of (P), hence these altitudes are to each other as 2:1 (9). Thus

$$\text{vol. } A' - BCD = 1/2 \text{ vol. } A - BCD = 1/2\ V,$$

where V is the volume of $ABCD$. Similarly we have

$$\text{vol. } A' - BCD = \text{vol. } C' - DAB = \text{vol. } D' = ABC = 1/2 \text{ V}.$$

Now if from (P) we cut away the four tetrahedrons $A' - BCD$, $B' - CDA$, $C' - DAB$, $D' - ABC$, what is left is precisely the tetrahedron $ABCD$, hence the

THEOREM. *The volume of a tetrahedron is equal to one-third of the volume of its circumscribed parallelepiped.*

19. Definitions. The common perpendicular of a pair of opposite edges of a tetrahedron may be called a "bialtitude" of the tetrahedron. A tetrahedron thus has three bialtitudes, one relative to each pair of opposite edges.

The bialtitudes of a tetrahedron are equal and parallel to the common perpendiculars of the three pairs of opposite faces of the parallelepiped circumscribed about the tetrahedron.

The angle between two skew lines is, by definition, the angle formed by the two parallels to these lines through a point in space.

20. THEOREM. *The volume of a tetrahedron is equal to one-sixth of the product of a pair of opposite edges times the corresponding bi-altitude times the sine of the angle between these two edges.*

The proposition is due to F. J. Servois (1767-1847), another distinguished pupil of Monge. It keeps on recurring in the mathematical journals, with various proofs. It is readily established by the use of the circumscribed parallelepiped.

The volume of (P) (Fig. 1) is equal to the altitude d relative to the face $A'BD'C$ multiplied by the area of this face, which area is equal to

$$1/2\ BC.A'D'.\text{Sin}\ (BC,\ A'D')$$

where the parenthesis denotes the angle between the diagonals BC, $A'D'$. Now AD is equal and parallel to $A'D'$, hence the volume of (P) is equal to

$$1/2\ d.BC.AD.\text{Sin}\ (BC,\ AD)$$

But d is equal to the bialtitude relative to BC, AD, and the volume of $ABCD$ is equal to one-third of the volume of (P) (18), hence the proposition.

21. The above result (20) leads to a very curious consequence. If the *two skew lines* BC, AD remain fixed in position, the angle (BC, AD) and the line of shortest distance d will remain the same, no matter where the segments BC, AD, are placed on these skew lines, hence the

THEOREM. *If two segments, of fixed length, slide along two fixed skew lines, in any way whatever and independently of one another, the volume of the variable tetrahedron determined by their four ends is constant.*

The proposition is due to the Swiss mathematician Jakob Steiner (1796-1863).

22. This property (21) may however be proved without the formula for the volume of the tetrahedron (20).

Let AD, $A'D'$ (Fig. 3) be any two positions of the segment AD on the first fixed skew line, and BC, $B'C'$ any two positions of the segment BC on the other given skew line.

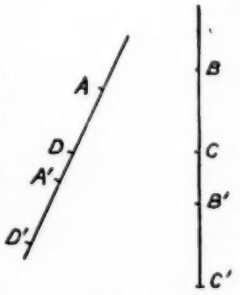


FIG. 3

The two triangles DBC, $DB'C'$, are equivalent, for their bases BC, $B'C'$ are equal and

their altitudes issued from D are identical, since the bases lie on the same line. Hence the two tetrahedrons $A - DBC$, $A - DB'C'$ are equivalent, for their altitudes issued from A are identical, the two bases being coplanar.

In a similar way we may show that the two tetrahedrons $B' - C'AD$, $B' - C'A'D'$ are equivalent. But $A - DB'C'$, $B' - C'AD$ are identical, hence $ABCD$ and $A'B'C'D'$ are equivalent, which proves the proposition.

23. If (P) (Fig. 1) is a cube, the inscribed tetrahedron (T) is obviously a regular tetrahedron. More generally, if (P) is any special parallelepiped, the properties of (P) will be reflected by (T). What kind of a tetrahedron is (T), if (P) is right and rectangular, if each face of (P) is a rhombus, etc.? For lack of space we shall leave the reader to answer these questions by his own devices, at least for the present.

Christiaan Huygens

April 14, 1629
June 8, 1695

CHRISTIAAN HUYGENS was the son of a man who has been called "the most brilliant figure in Dutch literary history," and he himself is characterized by Cajori as being "the prince of philosophers in Holland, and one of the greatest scientists of the seventeenth century." His training was in law and in mathematics. His chief work, as we shall see, was in mathematics, astronomy, and physics. His father had spent some time in England in the reign of James I: Huygens went to London in 1663 when he was made a member of the Royal Society, and again in 1689 following the publication of Newton's *Principia*. In 1665-66 Huygens went to Paris at the invitation of the king but he returned to Holland in 1681 when the revocation of the Edict of Nantes removed the measure of protection which protestants had enjoyed in France for nearly a century.

Huygen's work in mathematics and physics shows the interesting condition of things in the seventeenth century when the invention of new tools made many investigations possible and when the new methods of mathematics led to the formulation of the laws of physics. In 1651, Huygens published a criticism of a work on quadratures by Gregory

St. Vincent. He followed this by tracts on the quadrature of various curves including the cissoid. In 1654, he invented a new method of grinding and polishing lenses improving the telescope to such an extent that the rings of Saturn were revealed as rings and not as a pair of planets as had been previously supposed. This work in astronomy led him to the study of clocks for he needed a more accurate timepiece than the clocks then in use. Accordingly, he investigated the laws of the pendulum and he invented the pendulum clock in 1656.

He studied the properties of logarithmic curves, the catenary, the cissoid, and the tractrix, and wrote the first formal treatise on probability. It will be remembered that the pioneer work of Pascal and Fermat was given in a series of letters interchanged between the two investigators.

One of his most noteworthy contributions to physics was his theory of light. The Greeks had believed that the eye sent out something that might be said to feel the object seen,—that is, the sensation of light originates within the eye itself. Newton believed that the object seen sent out material particles which caused the sensation of sight by impinging upon the eye. Huygens and Hooke, an English contemporary of Newton, believed that light was due to a wave motion in the ether. For years, Newton's corpuscular theory was the one held by scientists. The work of Young and Fresnel in the nineteenth century brought them to Huygen's theory. The present state of the matter seems to be that certain phenomena bear out the wave theory and others confirm the corpuscular theory. Thus, the Newtonian theory was held for about a hundred years, then the wave theory was adopted and kept the stage for approximately another hundred years and to quote Professor Dayton C. Miller, "Whether a corpuscular theory, an ether theory, either one or both combined, will prevail when a Newton tercentenary is celebrated, one would not now venture to assert."* At any rate, the wave theory of light led Huygens to the invention of the achromatic eyepiece for the telescope which not only greatly improved the instrument itself, but which greatly reduced the eyestrain of the observer.

The roster of the achievements of Christiaan Huygens justifies Newton's name for him "Summus Hugenius."

VERA SANFORD

* "Newton and Optics." *Sir Isaac Newton 1727-1929*, Baltimore, 1928. p. 44.

The National Council of Teachers of Mathematics, Fourteenth Annual Meeting

Minneapolis, Minn.

February 24th and 25th, 1933

Nicollet Hotel

General Theme

REMOVING THE OBSTACLES TO PROGRESS

THE NATIONAL COUNCIL of Teachers of Mathematics is a national organization of mathematics teachers in elementary and secondary schools. Its purpose is the promotion and stimulation of better teaching of mathematics. The National Council operates chiefly through three divisions of its organization; namely, THE MATHEMATICS TEACHER, *The Year Book*, and the annual meeting of its members and board of directors.

The National Council is supported and maintained by direct membership of individuals, but desires and encourages the affiliation of other groups of teachers having kindred aims. The annual dues are $2.00 and THE MATHEMATICS TEACHER is mailed to all members. You doubtless know someone who should join. Will you help your profession by calling attention to the Council and the value of membership?

To become a member send $2.00 to THE MATHEMATICS TEACHER, 525 W. 120th St., New York City.

Program

The sessions of the fourteenth annual meeting of the Council are to be held at the Hotel Nicollet and are open to members and their friends, without any formality, but all persons are requested to register their attendance with the clerk who will be stationed near the entrance to the room where the meetings are held.

Friday, February 24

10:00 A.M.

Business Meeting of the Board of Directors

2:00 P.M.

Public Meeting—Preliminary Reports of Standing Committees

1. Examinations
   Chairman, Professor Schlauch
2. Geometry
   Chairman, Professor Ralph Beatley
3. Individual Differences
   Chairman, Professor Raleigh Schorling
   Chairman of Administrative Sub-Committee, Dr. C. N. Stokes
4. Policy
   Chairman, Professor J. O. Hassler

6:00 P.M.

Informal Supper Meeting of directors and delegates of affiliated associations and branches

8:00 P.M.

Address of Welcome—Mr. C. R. Reed, Superintendent of Schools, Minneapolis, Minn.

Address—"The Story of Mathematics and Its Future Development." Professor U. G. Mitchell, University of Kansas.

Saturday, February 25

9:00 AM.

Business Meeting of the National Council of Teachers of Mathematics. Open to All. Reports, Resolutions, and Election of officers.

10:00 A.M.

Central Topic: "The Problem of Examinations and of Drill."

1. From the Standpoint of the College Entrance Examination Board.
   Professor L. H. Whitcraft, Ball Teachers College, Muncie, Indiana.
2. From the Standpoint of the Newer Psychology.
   Professor R. H. Wheeler, University of Kansas.

Discussion led by Professor E. B. Lytle, University of Illinois.

2:00 P.M.

Central Topic: "The Ideal Preparation of a Teacher of Secondary Mathematics."

1. From the Standpoint of the Educator.
   Professor W. C. Bagley, Teachers College, Columbia University.
2. From the Standpoint of the Mathematician.
   Professor N. J. Lennes, University of Montana.

Discussion led by Professor W. D. Reeve, Teachers College, Columbia University.

6:30 P.M.

Banquet.

Address—"Mathematics and Its Applications." Professor G. A. Bliss, University of Chicago.

Affiliated associations and branches of the National Council are urged to send delegates to the annual meeting of the Council. It is essential that reservations of rooms be made at the earliest possible date at the Hotel Nicollet.

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A Tentative Program for the Sub-Committee on Administrative Phases of the Individual Differences Problem

C. N. Stokes
Teachers College, Temple University, Philadelphia
Joseph B. Orleans
George Washington High School
New York City

The following is a tentative outline of the work to be done by the sub-committee. Suggestions and criticisms are invited from the readers of The Mathematics Teacher. Assistance is also asked of those who are interested in the study. It is hoped that sufficient numbers of people from large and small institutions express willingness to cooperate so that a strong and representative sub-committee may be formed.

A. Organization of Content
- I. Courses of Study
  - a. State
  - b. City
- II. Subject Organizations
  - a. Integrated
  - b. Project
  - c. Unit
  - d. Contract
  - e. Daily Lesson
  - f. Goals
- III. Differentiations where Pupils are Grouped According to Ability
  - a. Under General Types of Group Instruction
  - b. Under Various Supervised Study Techniques
  - c. Under Special Forms of Individualized Instruction
- IV. Differentiations where Pupils are not Grouped According to Ability
  - a. Under General Types of Group Instruction

b. Under Various Supervised Study Techniques
c. Under Special Forms of Individualized Instruction

V. Motivation
a. In Text Books
b. Classroom Devices

B. Ability Grouping

I. Various Types and Their Evaluations
a. X. Y. Z. Grouping
b. .....
c. .....

II. Criteria for Grouping and Their Evaluations
a. I.Q., C.A., Previous Achievement
b. .....
c. .....

III. Roster Adaptations
a. To Provide for Shifting
b. .....
c. .....

C. Adaptations of Methods

I. Where Pupils are Grouped According to Ability
a. Various Types of Group Instruction
b. Various Types of Supervised Study
c. Various Types of Pure Individualization
d. Oral Methods

II. Where Pupils are not Grouped According to Ability
a. Various Types of Group Instruction
b. Various Types of Supervised Study
c. Various Types of Pure Individualization
d. Oral Methods

III. Evaluation of Methods
a. Immediate Recall
1. Differences in
2. Nature of Distributions
3. Relation to Intelligence
b. Delayed Recall
1. Differences at Interval from 3 mo. on
2. Amount Lost for Intervals from 3 mo. on
3. Nature of Distributions of Recall and Forgetting

4. Relative Amounts Lost
5. Types of Abilities Lost
6. Types of Abilities Retained

IV. Rate of Learning
- a. Differences in
- b. Relation to Intelligence

V. Relearning
- a. Rate at Various Intervals
- b. Permanence of Learning after Intervals of Relearning
- c. Appropriate Time for Relearning in Preparation for Sequence Material

VI. Motivation
- a. Individualization vs Classroom Procedures

D. Tests

I. Types
- a. Kinds and their Uses
- b. Methods of Construction
- c. Reliability and Validity

II. Test Items
- a. Appropriate Forms
- b. Validity

OFFICIAL FINAL BALLOT

For the Election of Officers at the Minneapolis meeting, February 24, 1933, of the National Council of Teachers of Mathematics.

For Second Vice-President, 1933-35 (Vote for One)

☐ BEATLEY, RALPH
Harvard University

☐ MILLER, A. BROWN
Cleveland, Ohio

For Members of the Board of Directors, 1933-36 (Vote for Three)

☐ BARBER, HARRY C.
Boston, Mass.

☐ FARNAM, LAURA M.
Minneapolis, Minn.

☐ HASSLER, J. O.
Norman, Okla.

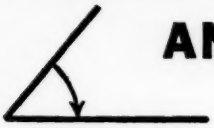
☐ KELLY, MARY
Wichita, Kan.

☐ SCHLAUCH, W. S.
New York City

☐ WALLIS, WILLIAM
Washington, D.C.

Please mark this ballot at once and mail same to Edwin W. Schreiber, Secretary, 619 W. Adams Street, Macomb, Illinois. Every ballot must be enclosed in a separate envelope with the name and address of the voter on the envelope. The ballots sent by mail must reach Macomb by February 20. Opportunity for voting will be given to bona fide members attending the annual meeting at the time of registration. The polls will close at 8:00 P.M. Friday, February 24. You must use this official printed ballot or send to the secretary for an official ballot.

PLANE TRIGONOMETRY AND LOGARITHMS



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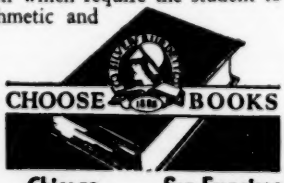

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